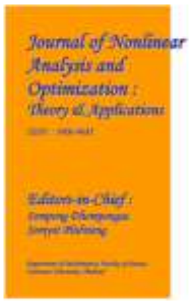
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 16, Issue. 1, No.1 : 2025
ISSN :**1906-9685**



METHODS FOR SOLVING DIFFERENTIAL EQUUATIONS

Nagaveni B, Research Scholar, P. K. University, Shivpuri, MP, India
Dr. Manoj Sharma, Professor, P.K. University, Shivpuri, MP, India

Abstract
This research paper provides a comprehensive summary of the techniques and methods used for solving second-order ordinary differential equations that have constant coefficients. We start with the introduction of the general form of a 2nd order differential equation and the explanation of the concept of constant coefficients. Furthermore, the paper provides the attributes of the equation and its roots that are used to show the nature of the solutions. Three possible cases are discussed i.e. distinct and real roots, complex conjugate roots. Then the general solution for each case is discussed, supported by examples for illustrating the various applications of the method. The conclusion further includes discussion of applications of second-order differential equations in the field of physics and engineering.

Keywords
- Ordinary Differential Equations
- Numerical Solution of Equations
- Exact Solution of Equations
- Taylor's Method
- Euler's Method
- Runge-Kutta Method
- Second-order differential equations
- Constant coefficients
- Homogeneous linear equations

Introduction
Numerical solution methods of ordinary differential equations in numerical analysis are important topics, that are usually used for many differential equations that is difficult to find their exact and analytic solution or the equation which cannot be represented in explicit form. There are many methods of numerical solution of ordinary differential equations such as; Taylor method, Euler method, Hunn method and Runge-Kutta method with first, second, third, fourth and higher orders respectively. Taylor's method is very accurate for numerical solution of differential equations, but it is rarely used because of the need for computations of successive derivatives. Euler's method has more errors but needs less computation. The Runge-Kutta method is a suitable and the most commonly used method with less computational steps and accurate calculation. The Runge-Kutta method is the generalized form of the Euler method which is used for numerical solution of ordinary differential equations. In this paper, the numerical solutions of ordinary differential equations are solved by Taylor, Euler and Runge-Kutta fourth-order methods and then their exact solutions are compared using tables and graphs.
Ordinary differential equations of second order, whose coefficients remain constant throughout the equation are a class of equations that frequently arise in many branches of mathematics and physics, making them of fundamental importance in understanding and predicting physical phenomena. These

equations have a wide range of applications in areas such as mechanics, electromagnetism, and quantum mechanics [1]. Therefore, finding the solutions to these equations is an essential task for many researchers and students in various fields of study. In this article, we aim to provide a comprehensive and detailed overview of the methods used to solve this class of equations. We begin by introducing the Standardized format of the second-order differential equation that has coefficients that remain constant throughout the equation, which is a homogeneous linear equation. We explain the meaning of constant coefficients and why they play an important role in solving these equations. We then present the characteristic equation and its roots, which provide information about the nature of the solutions. We discuss the three possible cases: real and distinct roots, complex conjugate roots, and repeated roots. For each case, we derive the general solution and
A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realize that computing machines are now readily available which reduce numerical work considerably.

Related Word

One of the earliest works on this topic was by Leonhard Euler in the 18th century. Euler developed a general method for solving LDE (linear differential equations) of any order, including the 2nd order equations coefficients with constant (Euler, 1748). This method involved finding the solutions or Int. J. Sci. Res. in Mathematical and Statistical Sciences Vol.10, Issue.2, Apr 2023 © 2023, IJSRMSS All Rights Reserved 10 values that satisfy the characteristic equation and using them to derive the general solution [3]. In the 19th century, the study of differential equations experienced significant developments, and many mathematicians contributed to the understanding of these equations. One of the most notable works in this period was by Carl Friedrich Gauss, who introduced the system of undetermined coefficients for solving LDE of any order (Gauss, 1815). This method was later applied to second-order equations with constant coefficients, providing a straightforward approach to finding particular solutions [4]. In the 20th century, the study of differential equations continued to grow, and new techniques were developed to solve these equations. One of the most significant developments in this period was the Laplace transform, which allowed solving differential equations in a different domain (i.e., frequency domain) (Churchill and Brown, 1990). The Laplace transform has been extensively used to solve 2nd order DE coefficients with constant, providing an efficient and powerful approach to finding the solution [5]

Literature Review

For finding numerical solution of ordinary differential equation in general, each numerical method has its own advantages and disadvantages of use [4]. Taylor’s method is one of the best methods and have proper accuracy but rarely used because of the need successive derivatives calculations [5]. Runge–Kutta methods have been presented for the integration of linear systems of ordinary differential equations with constant coefficients. when the step size is limited by stability, then the fourth-order method is the most suitable [6]. Runge-Kutta method and Usmani Agarwal method are compared with a new method for numerical solution of three problems and the result shows that in all three problems with step size of 0.1=h and 0.05=h the accuracy of new method is more than Usmani Agarwal and Runge Kutta methods but with step size of h=0.2 Usmani Agarwal method has more accuracy than the new method [7]. Numerical methods for systems of first order ordinary differential equations are tested on a variety of initial value problems. In this case Runge-Kutta methods are not competitive, but fourth or fifth order methods of this type are best for restricted classes of problems in which function evaluations accuracy requirements are not very stringent [8]. Numerical solution of linear and nonlinear equations is compared with Adomian decomposition and Runge-Kutta methods and the result shows that Adomian decomposition method is very powerful [9]. Adams-Moulton and Runge-Kutta-Merson Methods which are used for solving initial-value problems in ordinary differential equations are improved in case of efficiency by the Modified

Taylor method based on three derivatives [10]. Taylor's method is accurate but it is less commonly used because of its successive derivatives computation. The Euler's method is also a suitable method, but the error is more in this method. The Runge-Kutta method has different order and is more accurate than other methods and has less error.

Solution of a Differential Equation

The solution of an ordinary differential equation means finding an explicit expression for *y* in terms of a finite number of elementary functions of *x*. Such a solution of a differential equation is known as the closed or finite form of solution. In the absence of such a solution, we have recourse to numerical methods of solution.

Let us consider the first order differential equation

$dy/dx=f(x, y)$, given $y(x_0)= y_0$ (1)

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions either as a power series in *x* from which the values of *y* can be found by direct substitution, or a set of values of *x* and *y*. The methods of Picard and Taylor series belong to the former class of solutions. In these methods, *y* in (1) is approximated by a truncated series, each term of which is a function of *x*. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as single-step methods.

The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth, etc. belong to the latter class of solutions. In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations until sufficient accuracy is achieved. As such, these methods are called step-by-step methods. Euler and Runga-Kutta methods are used for computing y over a limited range of x- values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values. Therefore, Milne and Adams methods require starting values which are found by Picard's Taylor series or Runge-Kutta methods.

Numerical Solution of Ordinary Differential Equation

Initial and boundary conditions.

An ordinary differential equation of the nth order is of the form

$F (x, y, dy/dx, d_2y/dx_2 \ldots\ldots\ldots d_ny/ dx_n)=0$

Its general solution contains *n* arbitrary constants and is of the

$(x, y, c_1, c_2 \ldots\ldots\ldots c_n) =0$

To obtain its particular solution, n conditions must be given so that the constants c_1, c_2, c_n can be determined.

If these conditions are prescribed at one point only (say:x_0), then the differential equation together with the conditions constitute an initial value problem of the nth order. If the conditions are prescribed at two or more points, then the problem is termed as boundary value problem. In this chapter, we shall first describe methods for solving initial value problems and then explain the finite difference method and shooting method for solving boundary value problems.

The numerical solution of ordinary differential equations (ODEs) involves approximating the solution of ODEs using computational methods. These methods are particularly useful when analytical solutions are difficult or impossible to obtain. Here is an overview of the key concepts, techniques, and methods used for solving ODEs .

1. Problem Formulation

An ODE can generally be written as:

$y'(t)=f(t,y), \qquad y(t_0)=y_0$

$y(t)$: The solution we want to approximate.

$f(t,y)$: A function defining the relationship between *t*, *y*, and *y'*.

$y(t_0)=y_0$ The initial condition.

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Different Types of Numerical Methods

The numerical analysts and Mathematicians used have a variety of tools that they use to develop numerical methods for solving Mathematical problems. The most important idea, mentioned earlier, that cuts across all sorts of Ma

thematical problems is that of changing a given problem with a 'near problem' that can be easily solved. There are other ideas that differ on the type of Mathematical problem solved. An Introduction to Numerical Methods for Solving Common Division

Here are the most common methods for solving ODEs:

Euler's Method: This is a simple first-order method. This was the oldest and simplest method originated by Leonhard Euler in 1768. It is the first order numerical method for solving ordinary differential equations with given initial conditions. This method is use to analyze a differential equation (D.E.) which uses the idea of linear approximation, where we use small tangent lines over a short distance to approximate the solution to an initial value problem. Consider a D.E. $y' = (f(x, y(x))$ where $y(x_0) = y_0$.

The formula to solve given D.E. using Euler's method is given by the formula:

$y_{n+1}=y_n+hf(t_n,y_n)$.

Improved Euler Method: Euler method gives best result when the functions are linear in nature. When functions are non-linear there occurs a truncation error. To remove this drawback Improved Euler's method is introduced. In this method instead of point the arithmetic average of the slopes at x_n and x_{n+1}(that is, at the end points of each subinterval) is used. This method based on two values of dependent variable y_{n+1} and predicted value by Euler method i.e.,

$Y_{n+1}=y_n+h.f(x_n,y_n)$

$Y_{n+1}=y_n+h/2[f(x_n,y_n)+f(x_{n+1},y_{n+1})$

This method is also known as Heun's method or second order Runge-Kutta method. The local error occurred in this method is proportional to the cube of step size and the global error occurred in this method is proportional to square of step size.

C. Runge-Kutta Methods: Higher-order methods that improve accuracy while maintaining stability.

Fourth-Order Runge-Kutta (RK_4) is the most commonly used:

$k_1=hf(t_n,y_n)$

$k_2=hf(t_n+h/2, y_n+k_1/2)$

$k_3=hf(t_n+h/2, y_n+k_2/2)$

$k_4=hf(t_n+h/2, y_n+k_3/2)$

Conclusion

Here we apply various method to obtained solution for differential Equation. Numerical analysis is mathematics which has developed efficient methods for obtaining numerical solutions to difficult mathematical problems. Most of the mathematical problems that arise in science and engineering are very difficult and sometimes impossible to solve properly. Thus, an approximation is very important to make solving a difficult mathematical problem easier. Due to extreme developments in computational technology, numerical approximation has become more popular and has become a modern tool for scientists and engineers. As a result, many scientific software has been developed (for example, Matlab, Mathematic, Maple etc.) to handle more difficult problems in an efficient and easy manner. One can get the result by just one command without knowing the details of numerical method. Thus, one may ask why we need to understand numerical methods when such software is in our hands. In fact, to use some standard software as an end user, there is no need for in-depth knowledge of numerical methods and their analysis in most of the cases.

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